Some aspects of the synchronization in coupled maps

Sandro E. de Souza Pinto,* José T. Lunardi, Abdala M. Saleh, and Antonio M. Batista
Grupo de Física Teórica, Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa
Av. Gal. Carlos Cavalcanti 4748. CEP 84032-900, Ponta Grossa, PR, Brazil
(Dated: February 8, 2008)

Through numerical simulations we analyze the synchronization time and the Lyapunov dimension of a coupled map lattice consisting of a chain of chaotic logistic maps exhibiting power law interactions. From the observed behaviors we find a lower bound for the size N of the lattice, independent of the range and strength of the interaction, which imposes a practical lower bound in numerical simulations for the system to be considered in the thermodynamic limit. We also observe the existence of a strong correlation between the averaged synchronization time and the Lyapunov dimension. This is an interesting result because it allows an analytical estimation of the synchronization time, which otherwise requires numerical simulations.

PACS numbers: 05.45.Ra, 05.45.Xt, 05.45.Jn

arXiv:nlin/0504015v1 [nlin.CD] 6 Apr 2005

Coupled Map Lattices (CML's) were introduced in the literature as suitable models to study spatiotemporal behavior of spatially extended dynamical systems. Fundamentally they are systems defined on a discrete space-time and possessing continuous state variables. In the last two decades such models have received a great and increasing deal of interest, being applied to several nonlinear phenomena including systems as diverse as physical, chemical and biological[1]. CML's share with real complex systems one of their most intriguing behavior, which is the possibility of synchronization. Such a phenomenon can be observed in a great variety of real systems, going from electronic circuits until physiological processes, for example [2, 3].

The model we are concerned here consists of a chain of N identical chaotic logistic maps [4], each one located at a definite site in a discrete space, and coupled between themselves through a power law interaction [5]. Our aim is to go a step further in the understanding the synchronization of this system through the analysis of the synchronization time and the Lyapunov dimension.

The model is defined as follows. At the (discrete) instant of time n the state, or amplitude, of the map located at the site i ($i = 1, 2, \cdots, N$) is denoted by the continuous variable $x_n^{(i)}$. The state of the whole lattice at time n will be given by the N-dimensional vector $\mathbf{x}_n = \left(x_n^{(1)}, x_n^{(2)}, \cdots, x_n^{(N)}\right)$. The time evolution of the system is given, in matrix form, by the following mapping

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) + \mathbf{I}\,\mathbf{F}(\mathbf{x}_n)\,,$$

where $\mathbf{F}(\mathbf{x}) = \left(f(x^{(1)}), f(x^{(2)}), \ldots, f(x^{(N)})\right)$, and $\mathbf{I}$ is an $N \times N$ matrix which specifies the coupling among the elemental maps. The function $f(x)$ characterizes the elemental map, which we assume here to be the fully chaotic logistic one $f(x) = 4x(1-x)$, with $x \in [0, 1]$. We always assume odd N and periodic boundary conditions. The initial state of each elemental map is randomly chosen. The power law coupling is given by

$$I_{ij} = \frac{\varepsilon}{\eta}\left\{\frac{1}{r_{ij}^{\alpha}}\left(1-\delta_{ij}\right) - \eta\delta_{ij}\right\}\,,$$

where I_{ij} are the matrix elements of $\mathbf{I}$ and $r_{ij} = |i-j|$ is the "distance" between sites i and j. The parameters ε and α give, respectively, the strength and the range of the interaction and $\eta = 2\sum_r^{N'} r^{-\alpha}$ is a normalization factor, with $N' = (N-1)/2$. The parameter α can assume any value in the interval $[0, \infty)$. The extreme cases $\alpha = 0$ and $\alpha \to \infty$ correspond, respectively, to *global* (mean field) and *local* (first neighbors) couplings. Here we restrict the parameter ε to the interval $[0, 1]$ in order to get all the individual map amplitudes into the interval $[0, 1]$. In a recent work, including one of us, it was proposed that the parameter ε could assume any nonnegative value [6]. Although both systems exhibit the same behavior when $0 \le \varepsilon \le 1$, our choice implies that the nonlinearity of the model arises *solely* from the nonlinearity of the elemental maps, while the choice in ref. [6] allows, for $\varepsilon > 1$, also nonlinearities arising from the $\texttt{mod}\,1$ operation in the coupling scheme.

Among the various characterizations of synchronization [2] we choose the following. A system is said to be in a *completely synchronized state* at time n if all the elemental maps have the same amplitude, i.e., $x_n^{(1)} = x_n^{(2)} = \cdots = x_n^{(N)} = x_n^{*}$. The subspace $\mathbf{S}$ of all these states (the diagonal of the whole state space of the system) will be called *invariant* if $\mathbf{x}_m \in \mathbf{S}$ implies that $\mathbf{x}_n \in \mathbf{S}$ for all $n > m$. In this case a *synchronized regime* starts when the system state is put on $\mathbf{S}$, which is then called the *synchronization subspace*. Given specific initial conditions, the minimum value of m for which the system goes into this subspace is identified as the *synchronization time* t_s. The necessary and sufficient condition for $\mathbf{S}$ to be invariant is that the sum $\sum_j I_{ij}$ be independent of i, i.e., the sum of all elements of each line of the matrix $\mathbf{I}$ must be the same for all lines [7]. Besides that, in

*Electronic address: sandroesp@uepg.br

order to hold $x_{n+1}^{(i)}$ in the interval $[0,1]$, for all n and i, it is necessary that this sum be restricted to the interval $[-1,0]$. In our case $\sum_j I_{ij}=0$ for all i and $\mathbf{S}$ turns out to be an invariant subspace.

If initially the lattice is not synchronized, it only will synchronize after some time m if the parameters α and ε assume values into a suitable domain, for each N, in the parameter space [8, 9]. The boundaries of this domain can be calculated analytically from the condition $\lambda_2 = 0$, where λ_2 is the second largest Lyapunov exponent of the system, which is the largest Lyapunov exponent transversal to the synchronization subspace [10]. These boundaries can be decomposed into an upper line, given by $\varepsilon_c^{'}(\alpha,N) = 3/2(1-b^{(N')}/\eta)^{-1}$, and into a lower one, given by $\varepsilon_c(\alpha,N)=1/2(1-b^{(1)}/\eta)^{-1}$, where $b^{(k)}=2\sum_{m=1}^{N'}\cos(2\pi km/N)m^{-\alpha}$ $(1\leq k\leq N)$ are the eigenvalues of the circulant matrix $\mathbf{B}=(\eta\varepsilon^{-1}\mathbf{I}-\eta\varepsilon\mathbb{1})$ [6, 9]. For instance, the domains for $N=5$ and $N=385$ are shown in Figure 1. The shown lines are only the lower ones, for both N; the upper lines do not appear because in the range of figure all values of $\varepsilon_c^{'}(\alpha,N)$ are greater than 1.0, which are outside the range of this parameter. We can observe two distinct regions for each N. For $N=385$, and if it does not start synchronized, the lattice synchronizes after some time only in Region (A), while for $N=5$ it synchronizes only in Regions (A) and (B).

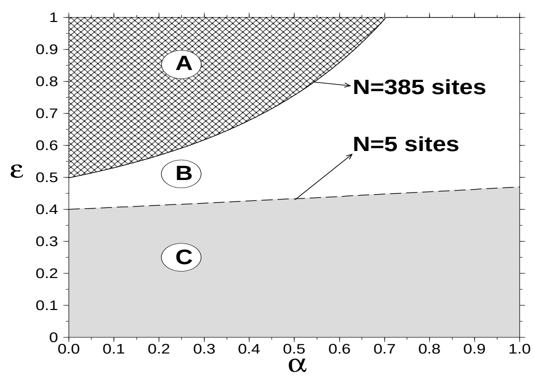

FIG. 1: Domains of synchronization in the parameters space. For $N=5$ the CML synchronizes in Regions (A) and (B) and does not in Region (C). For $N=385$ it synchronizes in Region (A) and does not in Regions (B) and (C).

On the state space of the system it can be defined the quantity $d_n=\sigma_n\sqrt{N}$, where σ_n is the standard deviation of the $x_n^{(i)}$ around the mean $\langle x_n^{(i)}\rangle$, with the averages taken with respect to the index i. It turns out that this quantity corresponds to the distance, in the state space, from the point $\mathbf{x}_n$ to the synchronization subspace $\mathbf{S}$ [11]. We thus use the condition $d_n=0$ as a diagnostic for the synchronization regime. Accordingly, the synchronization time t_s is identified with the lowest n for which $d_n=0$ [13].

Figure 2 is a plot of the averaged synchronization time

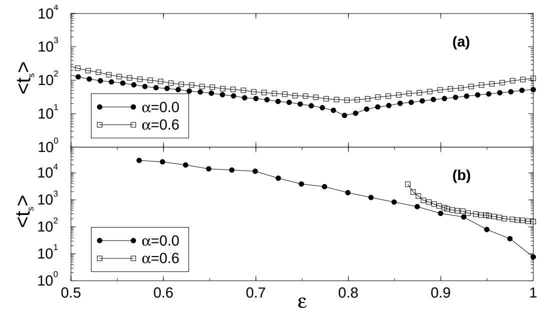

FIG. 2: Averaged synchronization time $<t_s>$ *versus* ε, with $\alpha=0.0$ and $\alpha=0.6$. (a) N=5; (b) N=385.

$\langle t_s\rangle$ *versus* the strength parameter ε, for two values of N and α [14]. For $N=5$ (Figure 2(a)), and for both the values of α, we observe an atypical behavior for the synchronization time when $\varepsilon>0.8$. It would be expected that $\langle t_s\rangle$ were always a decreasing function of the interaction strength, because greater values of this parameter would tend to facilitate synchronization. We have constructed similar plots for various values of N and α and our results indicate that this behavior is characteristic for small lattices, being therefore a finite size effect of the system. Such a behavior was not observed with respect to the α parameter. From our simulations we observed that the "turning point" $\varepsilon=\varepsilon^{(1)}$, above which $\langle t_s\rangle$ starts to increase, depends only on N (of course, with α inside the synchronization domain) and it can be clearly identified as

$$\varepsilon^{(1)} = \frac{\varepsilon_c'(0,N)-\varepsilon_c(0,N)}{2}. \qquad (1)$$

It turns out that the values of N for which this atypical behavior disappears must correspond to $\varepsilon^{(1)}\geq 1$. From the above expression (or from the numerical simulations) we can observe that this will be satisfied for $N\geq N_{min}=385$. The behavior of $\langle t_s\rangle$ for $N=N_{min}$ is shown in Figure 2(b).

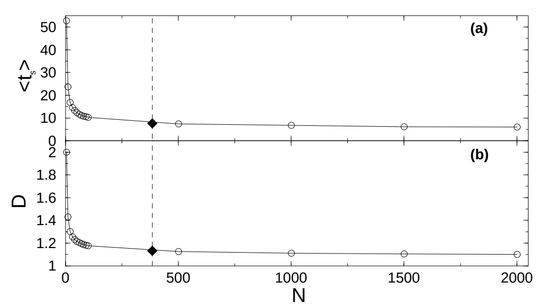

FIG. 3: (a) $\langle t_s\rangle$ *versus* N, for $\alpha=0.0$ and $\varepsilon=1.0$. (b) Lyapunov dimension D *versus* N, for the same values of α and ε. The vertical line corresponds to $N=385$.

In Figure 3 (a) we plot the averaged synchronization time with varying N, with both ε and α fixed. The results

indicate that $\langle t_s\rangle$ tends to saturate for large N. Moreover, the saturated time does not differs significantly from that corresponding to $N=385$.

Now we consider the CML in an outer vicinity of the synchronization domain in parameter space. The system does not more attain the synchronized regime if it not started synchronized. Figure 4(a) is a typical plot for a time series of the distances d_n in such a case. To make easier the visualization we presented our results in terms of $y_n = -\log_{10} d_n$. Figure 4(b) shows the corresponding statistical distributions of y_n. Again the results suggest that, for large lattices, such a distribution is practically independent of N and it is very well approximated by the distribution corresponding to $N=385$.

All the results presented so far indicate that $N=385$, apart from eliminating the just mentioned finite size qualitative effects, also lead to a reasonable quantitative approximation to the behavior of the system for larger N values, at least in what concern the synchronization time behavior and the statistical distribution of the distances d_n. So, at least in what concerns these two aspects, we claim that N_{min} sets a practical lower bound in numerical simulations for the system to be considered at the thermodynamic limit. With this claim we mean that both the qualitative and quantitative behaviors of the system at the thermodynamic limit $N\to\infty$ can be reasonably well approximated by the corresponding behavior of the system with $N=N_{min}$.

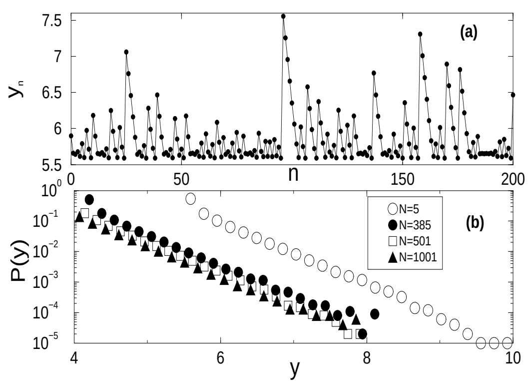

FIG. 4: (a) Time series for y_n with $N=385$, $\alpha = 0.6+10^{-5}$ and $\varepsilon = \varepsilon_c(0.6, 1001)$. (b) Distributions of y for $N=5$ (open circles), $N=385$ (filled circles), $N=501$ (open squares), and $N = 1001$ (filled triangles), with $\alpha = 0.6+10^{-5}$ and $\varepsilon = \varepsilon_c(0.6, N)$.

Now, we recall that the time oscillation of d_n in Figure 4(a) is due to the coexistence of both stable and unstable Lyapunov exponents in the direction transversal to the invariant subspace **S** [8, 12]. Therefore, a closer examination of the Lyapunov spectrum could reveal some new aspects of the synchronization behavior. We thus consider the Lyapunov dimension of the system, which is a suitable concept to study the Lyapunov spectrum and is defined as follows. Let λ_j $(j = 1, 2, \cdots)$ denote the j-th largest Lyapunov exponent of the system and p be the largest integer for which $\sum_{j=1}^{p}\lambda_j$ is non negative. Then D is given by [4]

$$D = \begin{cases} 0 & \text{if there is no such } p \\ p + \frac{1}{|\lambda_{p+1}|}\sum_{i=1}^{p}\lambda_i & \text{if } p < N \\ N & \text{if } p = N \end{cases} \qquad (2)$$

In Figure 5 we depict the Lyapunov dimension D *versus* the strength parameter ε, for $N = 5$ and $N = 385$, and for two values of α. We can observe that D assumes a maximum $D_{max} = N$ for small values of ε. As ε enters into the synchronization domain, identified in the figure by vertical lines, we can observe two distinct behaviors. For large N, it monotonically decreases, but for small N there is a value $\varepsilon=\varepsilon^{(2)}$ above which D starts to increase. Figure 3 (b) shows the dependence of D with N, for α

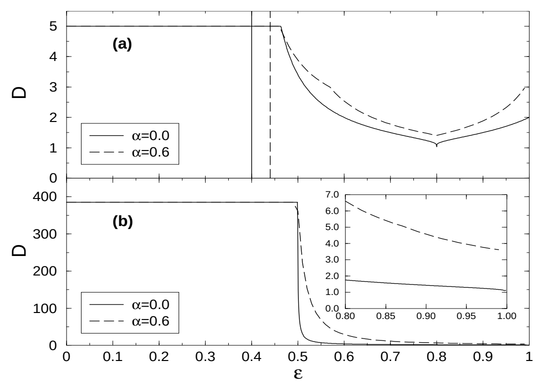

FIG. 5: Lyapunov dimension D *versus* ε with $\alpha = 0.0$ and $\alpha = 0.6$. (a) $N = 5$; (b) $N = 385$. The vertical lines indicate the boundary of the synchronization domain. *Inset*: detail of (b) for $0.8 \le \varepsilon \le 1.0$

and ϵ fixed within the synchronization domain. We can observe that the Lyapunov dimension tends to saturate with increasing N.

We point now the great similarity between the behavior of the Lyapunov dimension within the synchronization domain and the behavior observed for the averaged synchronization time $\langle t_s\rangle$, as can be seen by a direct comparison between the shapes of figures 2 and 5 or between figures 3 (a) and (b). These plots suggests that there is a correlation among the Lyapunov dimension and the averaged synchronization time. In Figure 6 we plot three dispersion diagrams $\langle t_s\rangle\times D$, each one with two of the three parameters α, ε, N fixed. All these diagrams give correlation coefficients ρ very close to 1, which indicates a *very strong* correlation among these two quantities. In these plots the dashed lines correspond to the fitting functions. In the first case the fitting is linear and in the two last they are exponentials. As the Lyapunov dimension can be analytically determined, this result allows also an analytical estimation of the synchronization time, which otherwise requires numerical simulations. The origin of such strong correlation between so diversely defined quantities is a point that would need a deeper analysis, which

is out the scope of this report. Nevertheless, we could try to understand this fact on some intuitive grounds by observing that it is reasonable to think that the dominance of negative(positive) Lyapunov exponents in the direction transversal to the invariant subspace **S** would tend to minimize(maximize) the synchronization time. Accordingly, as it is straightforward from its definition, this is precisely the behavior of the Lyapunov dimension.

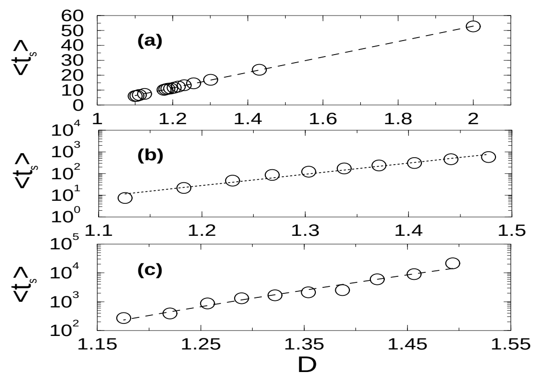

FIG. 6: Dispersion diagrams for $\langle t_s \rangle \times D$: (a) $\alpha = 0.0$, $\varepsilon = 1.0$ and N ranging from $N = 5$ to $N = 2000$, $\rho = 0.9998714$; (b) $\alpha = 0.0$, $N = 501$ and ε ranging from $\varepsilon = \varepsilon_c(0, 501)$ to $\varepsilon = 1.0$, $\rho = 0.9849924$; (c) $N = 501$, $\varepsilon = 1.0$ and α ranging from $\alpha = 0.0$ to $\alpha = 0.2$, $\rho = 0.9868354$.

Summarizing, in this report we numerically simulated the behavior of a CML consisting of a chain of chaotic logistic maps exhibiting power law interactions. We observed size dependent behaviors with respect to the averaged synchronization time $\langle t_s \rangle$ and with respect to the statistical distribution of the distances d_n, which allowed us to set $N = 385$ as a practical lower bound for this system to be considered in the thermodynamic limit in numerical simulations. We argued that the system behavior at the thermodynamic limit can be reasonably well approximated, both qualitatively and quantitatively, by its behavior at this lower bound. By the way, the behavior of systems exhibiting long range couplings, specially concerning their thermodynamical aspects, is still not well understood. We hope that these results could give some useful contribution to this subject.

The Lyapunov dimension for the system within the synchronization domain was studied and the results showed a very strong correlation among it and the averaged synchronization time. This result seems interesting because it allows an analytical estimation of the synchronization time, which otherwise requires numerical simulations. The origin of such a correlation and its related consequences are subjects that still need more clarifications and it will be postponed to future works. Finally, we remark that the statistical distribution of distances in Figure 2 can be well fitted by a power law. Further studies on this and other related scaling laws on this system are now in curse and shall be presented elsewhere.

Acknowledgments

The authors thank the supports from CNPq (SESP and AMB) and CNPq/Fundação Araucária (JTL). SESP and AMB also thank Ricardo L. Viana for fruitful discussions.

[1] K. Kaneko, *Theory and Applications of Coupled Map Lattices*, edited by K. Kaneko (Wiley, Chichester, 1993); P. M. Gade and C.-K. Hu, Physical Review E **60**, 4966, (1999); P. G. Lind, J. Corte-Real, J. A. C. Gallas, Phys. Rev. E **66**, 016219, (2002); J. W. Cain, E. G. Tolkacheva, D. G. Schaeffer, Daniel J. Gauthier, Phys. Rev. E **70**, 061906, (2004).

[2] A. Pikovsky, M. Rosemblum and J. Kurths, *Synchronization - A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge), (2001); S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares and C. S. Zhou, Phys. Rep. **366**, 1, (2002).

[3] Y. Mairesse, A. de Bohan , L. J. Frasinski et al. Science **302**, 1540, (2003); S. Yamaguchi, H. Isejima, T. Matsuo et al. Science **302**, 1408, (2003); S. Jalan, R. E. Amritkar, Phys. Rev. Lett. **90**, 014101, (2003); F. M. Atay, J. Jost, A. Wende, Phys. Rev. Lett. **92**, 144101, (2004).

[4] J. -P. Eckmann and D. Ruelle, Rev. Mod. Physics **57**, 617, (1985).

[5] A. Torcini and S. Lepri, Phys. Rev. E **55**, R3805 (1997); R. L. Viana and A. M. Batista, Chaos, Solitons and Fractals **9**, 1931, (1998); S. E. de S. Pinto and R. L. Viana, Physical Review E **61**, 5154, (2000).

[6] C. Anteneodo, A. M. Batista and R. L. Viana, Physics Letters A **326**, 227, 2004.

[7] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109, (1998).

[8] R. L. Viana, C. Grebogi, S. E. de S. Pinto, S. R. Lopes, A. M. Batista and J. Kurths, Phys. Rev. E **68**, 067204 (2003).

[9] C. Anteneodo, S. E. de S. Pinto, A. M. Batista and R. L. Viana, Phys. Rev. E. **68**, 045202(R) (2003) **erratum**; see also nlin.CD/0308014.

[10] P. M. Gade, H. A. Cerdeira and R. Ramaswamy, Phys. Rev E **52**, 2478 (1995).

[11] H. L. Yang and A. S. Pikovsky, Physical Review E **60**, 5474, (1999).

[12] R.L. Viana, S. E. de S. Pinto, C. Grebogi, Phys. Rev. E **66**, 046213 (2002).

[13] Due to the limited precision of numerical simulations, d is considered equals to zero if $d < 10^{-q}$, where q is the precision, which in our simulations is set to 16 (double precision).

[14] The average is taken with respect to a sample of 50 randomly chosen initial conditions.